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Exact solutions of double diffusive convection in cylindrical coordinates with $Le = 1$

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Abstract

The unsteady geometrical 2D governing equation set for the double diffusive convection—a very complicated nonlinear partial differential equation set with 4 variables—is solved analytically in the cylindrical coordinates. Two special exact solutions describing the convection in a cylindrical tube and a circular tube respectively are derived with an extraordinary method of separating variables and some other skills. The solutions are valuable for the development of heat and mass transfer theory. Moreover, as benchmark solutions, they are very useful for the computational heat and mass transfer to check the accuracy, convergence and effectiveness of various numerical computation methods.

Keywords: Analytical solution; Double diffusive convection; Heat and mass transfer; Nonlinear

1. Introduction

Double diffusive convection widely exists in a variety of practical applications and natural environment. In theoretical investigations of convection, analytical solutions are of significance. Many analytical solutions played key roles in the early development of fluid mechanics as well as the heat conduction [1,2]. However, the governing equations of double diffusive convection flow are nonlinear and coupled. Hence, it is highly difficult to obtain analytical solutions. To our knowledge, no new explicit analytical solutions of double diffusive convection flow has been reported except that recently derived by the first author in porous media [3].

Besides their theoretical meaning, analytical solutions can also be used to check the accuracy, convergence and effectiveness of various numerical computation methods and to improve their differencing schemes, grid generation ways and so on. The analytical solutions are, therefore, very useful even for the newly rapidly developing computational fluid dynamics and heat transfer. For example, several analytical solutions which can simulate the 3-D potential flow in turbomachine cascades were obtained by Cai et al. [4], and were successfully used to check the computational techniques and computer codes [4–7]. In addition, we have recently presented some explicit analytical solutions of unsteady nonlinear flow and heat transfer [8–21]. In this paper, algebraically explicit analytical solutions of unsteady double diffusive convection are derived to develop the theoretical understanding and to serve as the benchmark solutions for numerical calculations. The derivation procedure is mainly based on the method of separation variables with addition employed by the authors in previous researches. This method separates an unknown function $f(x, y)$ by assuming $f = X(x) + Y(y)$ instead of $f = X(x) \cdot Y(y)$ as done in common methods. By the way, the derivation procedure includes the trial and error method with the help of inspiration, experience and fortune since the governing equation set is very complicated. However, for a given analytical solution, its

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Nomenclature

C	dimensionless solute concentration	r_0	tube radius
C_0	constant	T_c	function of time for solute concentration
Gr_c	solutal Grashof number	T_θ	function of time for temperature
Gr_θ	thermal Grashof number	T_ω	function of time for vorticity
K_i	different constants	t	dimensionless time coordinate
Le	Lewis number	z	dimensionless axial coordinate
N	buoyancy ratio $N = Gr_c/Gr_\theta$	Z_c	function of z for solute concentration
n	constant	Z_θ	function of z for temperature
Pr	Prandtl number	Z_ψ	function of z for stream function
R_c	function of radius for solute concentration	Z_ω	function of z for vorticity
R_θ	function of radius for temperature	θ	dimensionless temperature
R_ψ	function of radius for stream function	ψ	dimensionless stream function
R_ω	function of radius for vorticity	ω	dimensionless vorticity
r	dimensionless radial coordinate		

correctness and suitability can be proven easily by substituting it into the governing equations.

2. Governing equations

The governing axisymmetric equations for the Newtonian and laminar binary fluid neglecting heat generation, viscous dissipation, chemical reaction and thermal radiation can be expressed as [22]

$$\frac{\partial\omega}{\partial t}+\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\omega}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\omega}{\partial z}-\frac{\omega}{r^2}\frac{\partial\psi}{\partial z} = \left(\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}+\frac{\partial^2\omega}{\partial z^2}-\frac{\omega}{r^2}\right)-Gr_\theta\left(\frac{\partial\theta}{\partial r}+N\frac{\partial C}{\partial r}\right), \quad (1)$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}=r\omega, \quad (2)$$

$$\frac{\partial\theta}{\partial t}+\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\theta}{\partial z}=\frac{1}{Pr}\left(\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}\right), \quad (3)$$

$$\frac{\partial C}{\partial t}+\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial C}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial C}{\partial z}=\frac{1}{PrLe}\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right). \quad (4)$$

The velocities are expressed as

$$u_z=-\frac{1}{r}\frac{\partial\psi}{\partial r},\ u_r=\frac{1}{r}\frac{\partial\psi}{\partial z}. \quad (5)$$

These equations are mathematic 3D partial differential equation set with four unknown dependent variables ω, ψ, θ and C. According to the knowledge of authors, no analytical solutions for this equation set have been given in open journals. In following chapters, two special exact solutions with evident physical meaning are derived to promote the development of double diffusive convection.

3. Derivation of explicit exact solutions

Since the main aim is to find possible analytical solutions but not solutions for given initial and boundary conditions, the derivation approach in this paper is different from the common method. We first find the possible solutions of Eqs. (1)–(5), and then decide what their initial and boundary conditions are. Such an approach is similar to the derivation of typical basic analytical solutions of incompressible fluid dynamics in early time.

According to the extraordinary method applied by the authors—the method of separating variables with addition, the governing equation set is simplified by

$$\omega = R_\omega(r)+Z_\omega(z)+T_\omega(t), \quad (6)$$

$$\psi = R_\psi(r)+Z_\psi(z), \quad (7)$$

$$\theta = R_\theta(r)+Z_\theta(z)+T_\theta(t), \quad (8)$$

$$C = R_c(r)+Z_c(z)+T_c(t). \quad (9)$$

Substituting abovementioned four simplified relations into governing equation set, it is obtained

$$T'_\omega+Z'_\psi R'_\omega/r-R'_\psi Z'_\omega/r-(R_\omega+Z_\omega+T_\omega)Z'_\psi/r^2 = R''_\omega+R'_\omega/r+Z''_\omega-(R_\omega+Z_\omega+T_\omega)/r - Gr_\theta R'_\theta - Gr_c R'_c, \quad (10)$$

$$R''_\psi+Z''_\psi-R'_\psi/r=r(T_\omega+R_\omega+Z_\omega), \quad (11)$$

$$T'_\theta+Z'_\psi R'_\theta/r-R'_\psi Z'_\theta/r=(R''_\theta+R'_\theta/r+Z''_\theta)/Pr, \quad (12)$$

$$T'_c+Z'_\psi R'_c/r-R'_\psi Z'_c/r=(R''_c+R'_c/r+Z''_c)/PrLe. \quad (13)$$

We first separate Eq. (11). It can be rearranged as

$$T_\omega = R''_\psi/r-R'_\psi/r^2-R_\omega+Z''_\psi/r-Z_\omega. \quad (11a)$$

Then T_ω has to be a constant

$$T_\omega = K_1. \quad (14)$$

If Z_ψ is a linear function

$$Z_\psi = K_2 z+K_3, \quad (15)$$

then Eq. (11a) can be separated further

$$K_1+Z_\omega = K_5 = R''_\psi/r-R'_\psi/r^2-R_\omega. \quad (16)$$

From Eq. (16), it is derived

$$Z_\omega = K_5-K_1, \quad (17)$$

and

$$R''_\psi - R'_\psi/r = f(r) = (R_\omega + K_5)r \tag{18}$$

where $f(r)$ is an arbitrary function of r.

If $f(r)$ is given, then R'_ψ and R_ω can be deduced with common method as

$$R'_\psi = r\int \frac{f(r)}{r}\mathrm{d}r - 2K_6 r, \tag{19}$$

and

$$R_\omega = f(r)/r - K_5. \tag{20}$$

Of course, not any $f(r)$ can satisfy the demand of deriving an explicit analytical solution. It is well-know that the integration $\int[f(r)/r]\mathrm{d}r$ of any arbitrary $f(r)$ cannot be integrated in an explicit analytical form. In addition, not any $f(r)$ can satisfy governing equation (10), and it will be illustrated with example later.

Now, we separate Eq. (12). Besides abovementioned assumption, we add following relations to satisfy the demand of separating variables

$$T_\theta = K_4 t, \tag{21}$$

and

$$Z_\theta = K_7 z + K_{11}. \tag{22}$$

Substituting Eqs. (15), (21) and (22) into Eq. (12), following equation for R_θ can be derived

$$R''_\theta + (1 - K_2 Pr)R'_\theta/r = Pr(K_4 - K_7 R'_\psi/r). \tag{23}$$

If R'_ψ is deduced from Eq. (19) with a given $f(r)$, R_θ can be obtained with Eq. (23). As mentioned before, not any R'_ψ [equivalent to $f(r)$] can be integrated in Eq. (23).

Now, we have given a preliminary expressions of ω, ψ and θ [Eqs. (6)–(9), (14), (15), (17), and (19)–(23)] satisfying governing equations (2) and (3) with an undetermined function $f(r)$.

In order to obtain exact solution, we only consider the case $Le = 1$. For such condition, the governing equation for C—Eq. (4) is totally the same as Eq. (3) for θ. Then, the expression of C will be

$$C = K_0\theta + C_0, \tag{24}$$

where K_0 is an arbitrary undetermined constant.

Then, solutions satisfying Eqs. (2)–(4) with arbitrary undetermined function $f(r)$ and constant K_0 have been given. They have to satisfy Eq. (1) also and be able to be integrated explicitly. It is done by trial and error method.

A simple example is given first. It is assumed that $f(r)$ is a simplest power function with only one term and its exponential n is an integer

$$f(r) = n(n-2)K_9 r^{n-2}. \tag{25}$$

If assuming $K_2 = 0$ further, it can be deduced from Eq. (19) that

$$R'_\psi = nK_9 r^{n-1} - 2K_6 r, \tag{26}$$

and then derived from Eq. (23) that

$$R_\theta = Pr[-K_7K_9r^n/n + (K_4 + 2K_6K_7)r^2/4 + K_8\ln r]. \tag{27}$$

From Eqs. (6)–(9), (14), (15), (17), and (19)–(27), the expression of ω, ψ, θ and C with $K_2 = 0$ satisfying governing equations (2)–(5) are as follows

$$\omega = n(n-2)K_9 r^{n-3}, \tag{28}$$

$$\psi = K_9 r^n - K_6 r^2 + K_3, \tag{29}$$

$$\theta = Pr[(K_4 + 2K_6K_7r^2/4) - K_7K_9r^n/n] + K_8\ln r + K_4 t + K_7 z + K_{11}, \tag{30}$$

and

$$C = K_0\theta + C_0. \tag{31}$$

These expressions have to satisfy governing equation (1). Substituting them into Eq. (1), it can be found that only when $n = 4$ or 2 and $K_0 = -Gr_\theta/Gr_c$ the requirement is satisfied. So an exact solution for governing equation set (1)–(5) is found as (The case of $n = 2$ only is omitted since it represents a simple solution with constant velocity)

$$\omega = 8K_9 r, \tag{32}$$

$$\psi = K_9 r^4 - K_6 r^2 + K_3, \tag{33}$$

$$\theta = Pr[(K_4 + 2K_6K_7)r^2/4 - K_7K_9r^4/4] + K_8\ln r + K_4 t + K_7 z + K_{11}, \tag{34}$$

and

$$C = -(Gr_\theta/Gr_c)\theta + C_0. \tag{35}$$

The physical meaning of this solution will be explained with figures in next paragraph. In brief, it can represent a double diffusive convection flow in an infinite long cylindrical tube (when $K_8 = 0$).

When

$$f(r) = 8K_9 r^2 + K_2K_{10}(K_2+2)r^{K_2}, \tag{36}$$

is assumed, in which $K_2 \neq 0$, an exact solution can be derived similarly by more complicated operation with abovementioned derivation procedure. With the same assumptions mentioned in Eqs. (6)–(9), (14), (15), (21), (22), and (24), following solution can be deduced

$$\omega = 8K_9 r + K_2K_{10}(K_2+2)r^{K_2-1}, \tag{37}$$

$$\psi = K_9 r^4 + K_{10} r^{K_2+2} - K_6 r^2 + K_2 z + K_3, \tag{38}$$

$$\theta = Pr(K_4 + 2K_6K_7)r^2/[2(2 - K_2Pr)] + K_8 r^{K_2Pr}/(K_2Pr) - K_7Pr[K_9r^4/(4 - K_2Pr) + K_{10}r^{K_2+2}/(K_2 + 2 - K_2Pr)] + K_4 t + K_7 z + K_{11} \tag{39}$$

and

$$C = -(Gr_\theta/Gr_c)\theta + C_0. \tag{40}$$

By the way, K_2 in this solution is a real number and need not to be an integer.

The physical description of this solution can be a double diffusive convection flow in an infinite long circular tube with porous wall. It will be explained in next paragraph too.

By the way, both previous solutions are mainly exponential functions $f(r)$ and derived with the method of separating variables with addition. However, when choosing some other function types as $f(r)$ or using common method of separating variables, explicit solutions with simple physical meaning and without infinite series or special function have not been yet able to be derived. Further research work is needed to find more solutions.

4. Physical description of the first exact solution

Different values of K_is and different relations between K_is would represent different physical situation.

The explicit analytical solution—Eqs. (32)–(35)—can represent a double diffusive convection in an infinite long cylindrical tube when $K_8 = 0$ (otherwise there will be infinite temperature at $r = 0$). From Eq. (5) and Eq. (33), it is obtained

$$u_r = 0 \tag{41}$$

and

$$u_z = -4K_9r^2 + 2K_6. \tag{42}$$

It means that there is only z-direction convection flow in an infinite long cylindrical tube with parabolic axial velocity distribution. Considering that the velocity on the wall has to be zero for viscous fluid flow, the tube radius r_0 has to be considered that $u_z = -4K_9r_0^2 + 2K_6 = 0$, i.e.

$$r_0 = \sqrt{K_6/2K_9}. \tag{43}$$

When $K_6 = 2$ and $K_9 = 1$, the velocity distribution is shown in Fig. 1.

The temperature is unsteady if $K_4 \neq 0$ and is a linear increasing or decreasing function of time for $K_4 > 0$ or $K_4 < 0$. The axial temperature distribution is linear also, when K_7 is larger or less than zero, the temperature will be higher or lower along axial direction. The radial temperature distribution is a little bit more complicated but still rather simple, for example, for the steady solution ($K_4 = 0$) as well as $K_6 = -2$, $K_7 < 0$, $K_8 = 0$, $K_9 = -1$ and $K_{11} > -K_7(0.75Pr + 1)$, and the radial temperature distribution on $z = 1$ plane is shown in Fig. 2.

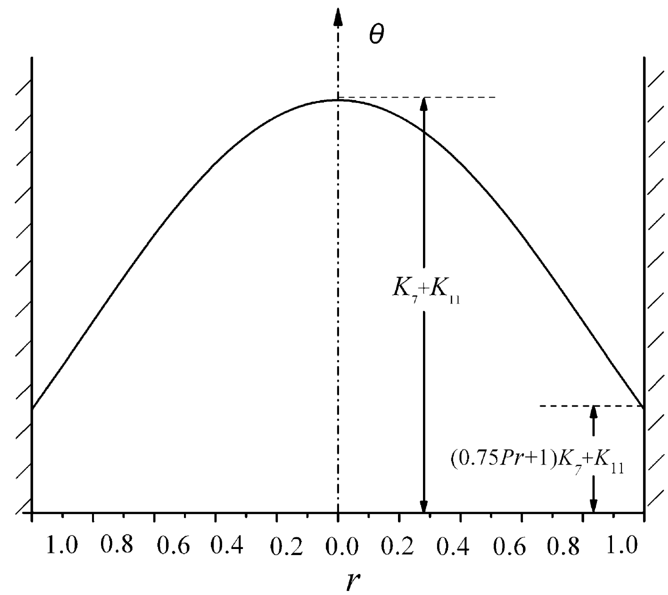

Fig. 2. The temperature distribution of Eq. (34) with $K_6 = -2$, $K_7 < 0$, $K_8 = 0$, $K_9 = -1$, $K_{11} > -K_7(0.75Pr + 1)$ and $z = 1$.

The distribution of solute concentration is completely similar to the distribution of temperature but upside down. The distribution of vorticity is a linear function of radius. It means that the flow is not a potential one but there is one vorticity only on the $z =$ Const. plane.

5. Physical description of the second exact solution

The explicit analytical solution—Eqs. (37)–(40)—with $K_2 \neq 0$ can represent a double diffusive convection in an

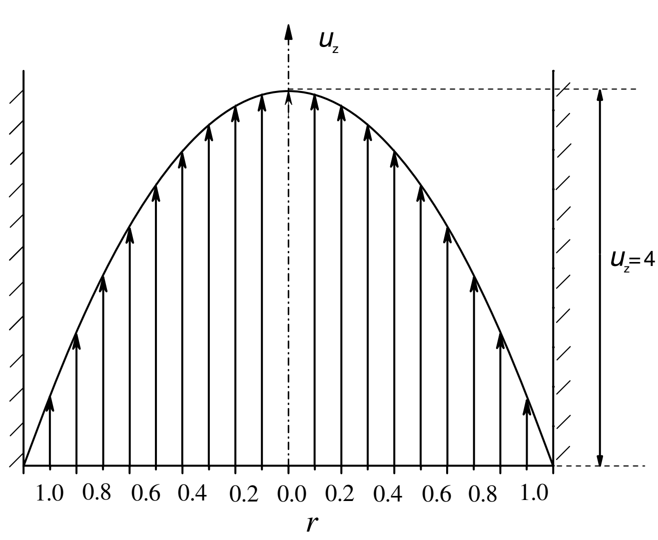

Fig. 1. The velocity distribution of solution (33) with $K_6 = 2$ and $K_9 = 1$.

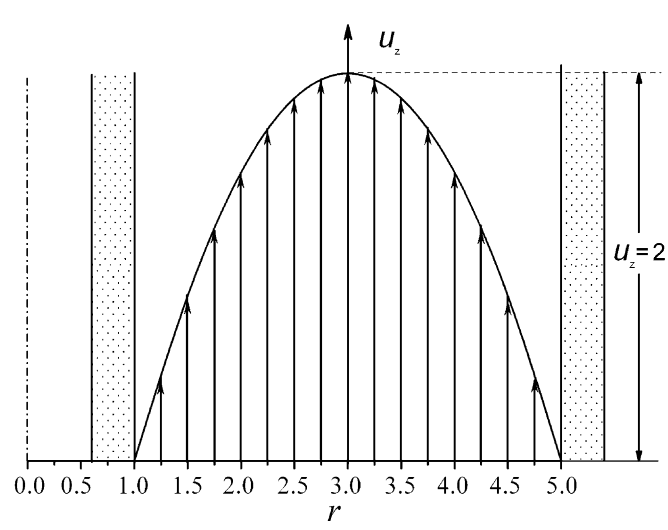

Fig. 3. The axial velocity distribution of solution (38) with $K_2 = 1$, $K_6 = -1.25$, $K_9 = 0.125$, $K_{10} = -1$.

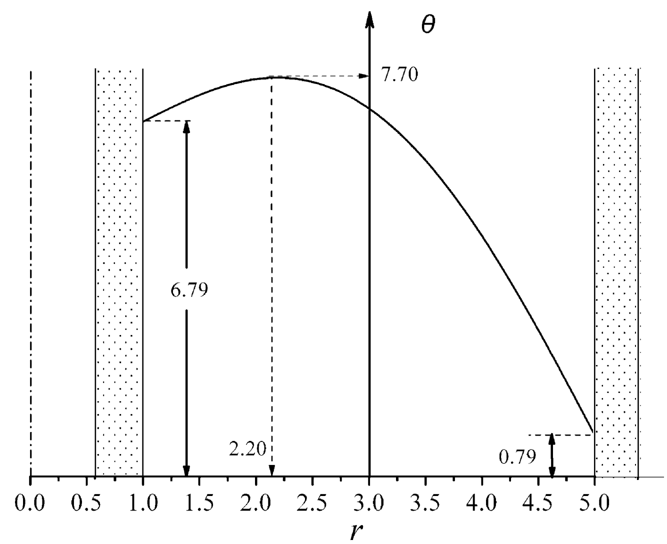

Fig. 4. The temperature distribution of Eq. (39) with $K_2 = 1$, $K_4 = 0$, $K_6 = -1.25$, $K_7 = -1$, $K_8 = 0$, $K_9 = 0.125$, $K_{10} = -1$, $K_{11} = 7$, $Pr = 1$ and $z = 1$.

infinite long circular porous tube. As a simple example, we choose $K_2 = 1$, then the expressions of radial and axial velocity are as following

$$u_r = K_2/r \tag{44}$$

and

$$u_z = -(4K_9r^2 + 3K_{10}r - 2K_6). \tag{45}$$

We consider the axial velocity expression first. There are two radiuses with zero axial velocity

$$r_{1,2} = \left[-3K_{10} \pm \sqrt{9K_{10}^2 + 32K_6K_9}\right]/(8K_9). \tag{46}$$

For some values of K_i, for example, when $K_6 < 0$, $K_9 > 0$, $K_{10} < 0$, and $9K_{10}^2 >| 32K_6K_9 |$, there will be two radiuses r_1 and r_2 larger than zero, then the inner and outer radius of the circular tube should be r_1 and r_2. The radial distribution of axial velocity is a parabolic curve too, similar to previous solution.

However, the radial velocity is a function of radius and cannot be zero at $r = r_1$ and $r = r_2$ or any r. Then, the wall of circular tube for this solution should be porous with media injecting into and ejecting from the tube wall.

The temperature distribution is a little bit more complicated than that of previous solution. However, it is still a polynomial function.

The axial velocity and temperature distribution along r are given in Figs. 3 and 4 respectively with $K_2 = 1$, $K_4 = 0$, $K_6 = -1.25$, $K_7 = -1$, $K_8 = 0$, $K_9 = 0.125$, $K_{10} = -1$, $K_{11} = 7$ and $Pr = 1$.

With different values of K_i, some other flow and thermal fields can be deduced, for example, more complicated fields with larger K_2 or K_i is not an integer.

6. Summary

Two algebraically explicit analytical solutions for the double diffusive convection have been derived. The governing equation set for such convection phenomena is rather complicated. It is a nonlinear mathematical 3-D equation set and has not been solved analytically yet. With an extraordinary method of separating variables—the method of separating variables with addition as well as with the help of inspiration, experience and future, we successfully derive exact solutions for two cases. One is for the double diffusive convection in an infinite long cylindrical tube. The other is for the double diffusive convection in an infinite long circular tube with porous wall. They are meaningful for the theory of heat and mass transfer. Especially, they can be benchmark solutions to check various performances of the codes of computational heat and mass transfer (CHT) and to improve various CHT methods and skills.

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